

# On the differentiability of Cauchy horizons\*

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## Abstract

Chruściel and Galloway constructed a Cauchy horizon that is nondifferentiable on a dense set. We prove that in a certain class of Cauchy horizons densely nondifferentiable Cauchy horizons are generic. We show that our class of densely nondifferentiable Cauchy horizons implies the existence of densely nondifferentiable Cauchy horizons arising from partial Cauchy surfaces and also the existence of densely nondifferentiable black hole event horizons.

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# 1 Introduction

Recently Chruściel and Galloway [4] have constructed an example of a Cauchy horizon which fails to be differentiable on a dense subset. In this paper we show that densely nondifferentiable Cauchy horizons appear to be generic in a certain class of Cauchy horizons. Chruściel and Galloway have also shown that their example implies the existence of a densely nondifferentiable black hole event horizon. They point out that these examples raise definite questions concerning some major arguments that have been given in the past where smoothness assumptions were implicitly made. In the light of these new examples, it is clear that there is a real need for a deeper understanding of the differentiability properties of horizons.

In a spacetime with a partial Cauchy surface  $S$  the Cauchy horizon  $H(S)$  is the boundary of the set of points where, in theory, one may calculate everything in terms of the initial data on  $S$ . Cauchy horizons are *achronal* (i.e., no two points on the horizon may be joined by a timelike curve) and this implies that Cauchy horizons (locally) satisfy a Lipschitz condition. This, in turn, implies that Cauchy horizons are differentiable almost everywhere. Because they are differentiable except for a set of (three-dimensional) measure zero, it seems that they have often been assumed to be smooth except for a set which may be more or less neglected. However, one must remember in the above that: (1) differentiable only refers to being differentiable at a single point, and (2) sets of measure zero may be quite widely distributed.

For  $S$  a closed achronal set each point  $p$  of a Cauchy horizon  $H^+(S)$  lies on at least one null generator [9]. However, null generators may or may not remain on the horizon when they are extended in the future direction. If a null generator leaves the horizon, then there is a last point where it remains on the horizon. This last point is said to be an *endpoint* of the horizon. Endpoints where two or more null generators leave the horizon are points where the horizon must fail to be differentiable [8], [4]. In addition, Chruściel and Galloway [4] have shown that Cauchy horizons are differentiable at points which are not endpoints. Beem and Królak have shown [5] that Cauchy horizons are differentiable at endpoints where only one generator leaves the horizon. These results give a complete classification of (pointwise) differentiability for Cauchy horizons in terms of null generators and their endpoints. Beem and Królak have also shown [5] that if we consider an open subset  $W$  of the Cauchy horizon  $H^+(S)$  and assume that the horizon has no endpoints

on  $W$ , then the horizon must be differentiable at each point of  $W$  and, in fact, that the horizon must be at least of class  $C^1$  on  $W$ . Conversely, the differentiability on an open set  $W$  implies there are no endpoints on  $W$ .

For general spacetimes, horizons may fail to be stable under small metric perturbations; however, some sufficiency conditions for various stability questions have been obtained [1], [6].

## 2 Preliminaries

**Definition 1** *A space-time  $(M, g)$  is a smooth  $n$ -dimensional, Hausdorff manifold  $M$  with a semi-Riemannian metric  $g$  of signature  $(-, +, \dots, +)$ , a countable basis, and a time orientation.*

A set  $S$  is said to be *achronal* if there are no two points of  $S$  with timelike separation.

We give definitions and state our results in terms of the future horizon  $H^+(S)$ , but similar results hold for any past Cauchy horizon  $H^-(S)$ .

**Definition 2** *The future Cauchy development  $D^+(S)$  consists of all points  $p \in M$  such that each past endless and past directed causal curve from  $p$  intersects the set  $S$ . The future Cauchy horizon is  $H^+(S) = \overline{(D^+(S))} - I^-(D^+(S))$ .*

Let  $p$  be a point of the Cauchy horizon; then there is at least one null generator of  $H^+(S)$  containing  $p$ . Each null generator is at least part of a null geodesic of  $M$ . When a null generator of  $H^+(S)$  is extended into the past it either has no past endpoint or has a past endpoint on  $\text{edge}(S)$  [see [9], p. 203]. However, if a null generator is extended into the future it may have a last point on the horizon which then said to be an *endpoint* of the horizon. We define the *multiplicity* [see [5]] of a point  $p$  in  $H^+(S)$  to be the number of null generators containing  $p$ . Points of the horizon which are not endpoints must have multiplicity one. The multiplicity of an endpoint may be any positive integer or infinite. We call the set of endpoints of multiplicity two or higher the *crease set*, compare [4]. By a basic Proposition due to Penrose [[9], Prop. 6.3.1]  $H^+(S)$  is an  $n - 1$  dimensional Lipschitz topological submanifold of  $M$  and is achronal. Since a Cauchy horizon is Lipschitz it follows from a theorem of Rademacher that it is differentiable almost everywhere (i.e. differentiable

except for a set of  $n - 1$  dimensional measure zero). This does not exclude the possibility that the set of non-differentiable points is a dense subset of the horizon. An example of such a behaviour was given by Chruściel and Galloway [4].

Following [5] let us introduce the notion of differentiability of a Cauchy horizon. Consider any fixed point  $p$  of the Cauchy horizon  $H^+(S)$  and let  $x^0, x^1, x^2, x^3$  be local coordinates defined on an open set about  $p = (p^0, p^1, p^2, p^3)$ . Let  $H^+(S)$  be given near  $p$  by an equation of the form

$$x^0 = f_H(x^1, x^2, x^3)$$

The horizon  $H^+(S)$  is *differentiable* at the point  $p$  iff the function  $f_H$  is differentiable at the point  $(p^1, p^2, p^3)$ . In particular, if  $p = (0, 0, 0, 0)$  corresponds to the origin in the given local coordinates and if

$$\Delta x = (x^1, x^2, x^3)$$

represents a small displacement from  $p$  in the  $x^0 = 0$  plane, then  $H^+(S)$  is differentiable at  $p$  iff one has

$$f_H(\Delta x) = f_H(0) + \sum a_i x^i + R_H(\Delta x) = 0 + \sum a_i x^i + R_H(\Delta x)$$

where the ratio  $R_H(\Delta x)/|\Delta x|$  converges to zero as  $|\Delta x|$  goes to zero. Here we use

$$|\Delta x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

If  $H^+(S)$  is differentiable at the point  $p$ , then there is a well defined 3-dimensional linear subspace  $N_0$  in the tangent space  $T_p(M)$  such that  $N_0$  is tangent to the 3-dimensional surface  $H^+(S)$  at  $p$ . In the above notation a basis for  $N_0$  is given by  $\{a_i \partial/\partial x^0 + \partial/\partial x^i \mid i = 1, 2, 3\}$ .

**Theorem 1** (*Chruściel and Galloway [4]*)

*There exists a connected set  $K \subset R^2 = \{t = 0\} \subset R^{2,1}$ , where  $R^{2,1}$  is a  $2 + 1$  dimensional Minkowski space-time, with the following properties:*

1. *The boundary  $\partial K = \bar{K} - \text{int } K$  of  $K$  is a connected, compact, Lipschitz topological submanifold of  $R^2$ .  $K$  is the complement of a compact set  $R^2$ .*

2. There exists no open set  $\Omega \subset R^{2,1}$  such that  $\Omega \cap H^+(K) \cap \{0 < t < 1\}$  is a differentiable submanifold of  $R^{2,1}$ .

**Proposition 1** (*Beem and Królak [5]*)

Let  $W$  be an open subset of the Cauchy horizon  $H^+(S)$ . Then the following are equivalent:

1.  $H^+(S)$  is differentiable on  $W$ .
2.  $H^+(S)$  is of class  $C^r$  on  $W$  for some  $r \geq 1$ .
3.  $H^+(S)$  has no endpoints on  $W$ .
4. All points of  $W$  have multiplicity one.

Note that the four parts of Proposition 1 are logically equivalent for an *open* set  $W$ , but that, in general, they are not necessarily equivalent for sets which fail to be open. Using the equivalence of parts (1) and (3) of Proposition 1, it now follows that near each endpoint of multiplicity one there must be points where the horizon fails to be differentiable. Hence, each neighborhood of an endpoint of multiplicity one must contain endpoints of higher multiplicity. This yields the following corollary.

**Corollary 1** ([5])

If  $p$  is an endpoint of multiplicity one on a Cauchy horizon  $H^+(S)$ , then each neighborhood  $W(p)$  of  $p$  on  $H^+(S)$  contains points where the horizon fails to be differentiable. Hence, the set of endpoints of multiplicity one is in the closure of the crease set.

### 3 A generic densely nondifferentiable Cauchy horizon

We shall construct a densely nondifferentiable Cauchy horizon in the 3-dimensional Minkowski space-time  $R^{2,1}$ , but our construction can be generalized in a natural way to higher dimensions. Let  $\Sigma$  be the surface  $t = 0$ , and let  $K$  be a compact, convex subset of  $\Sigma$ . Let  $\partial K$  denote the boundary of  $K$ . Let  $\rho(x, R)$  and  $D(x, R)$  be respectively a circle and a disc with center at  $x$  and radius  $R$ .

**Definition 3** A circle  $\rho(x, R)$  is internally tangent to the boundary  $\partial K$  of  $K$  if the disc enclosed by  $\rho$  is contained in  $K$  and for all  $\epsilon$  the disc of radius  $R + \epsilon$  and center  $x$  is not contained in  $K$ .

Let  $\rho(x, R)$  be internally tangent to  $\partial K$ ; then the point  $(x, R) \in R^{2,1}$  belongs to the future Cauchy horizon  $H^+(K)$  and conversely, if a point  $(x, R) \in R^{2,1}$  belongs to  $H^+(K)$  then the circle  $\rho(x, R)$  is internally tangent to  $\partial K$ . If  $\rho(x, R)$  is internally tangent in at least two points of  $\partial K$  then it follows from Proposition 1 that  $H^+(K)$  is not differentiable at the point  $(x, R)$  and the point  $(x, R)$  has multiplicity at least two.

We shall first construct a continuous curve that is not differentiable on any open subset. Let us take a line segment  $l_0$  and let us consider an isosceles triangle with base  $l_0$  and let  $\alpha_0$  be the angle at the base and let  $l_1$  denote the broken line consisting of two equal arms of the triangle. In the next step we construct two isosceles triangles with bases that are segments of the broken line  $l_1$  and we choose the angles  $\alpha_1$  at the base equal  $q \times \alpha_0$  where  $q < 1/2$ . We iterate the above construction. At the  $N$ th step of the construction the number of nondifferentiable points of the curve increases by  $2N - 1$ . After the  $N$ th step of the iterative procedure the vertex angle of the isosceles triangle obtained in the  $i$ th step is given by

$$\angle_N(x_i) = \pi - 2\alpha_1 \left[ q^{i-1} - \frac{q^i - q^N}{1-q} \right]. \quad (1)$$

In the limit  $N \rightarrow \infty$  the  $i$ th vertex angle is given by  $\pi - 2\alpha_1 q^{i \frac{q^{-1}-2}{1-q}}$  and is strictly less than  $\pi$  as  $q < 1/2$ .

Let us call the nowhere differentiable continuous curve constructed above a *rough curve*. Let us call a region of  $\Sigma$  that is bounded by a rough curve and two straight lines perpendicular to the rough curve at its two endpoints<sup>1</sup> a *fan*. The above construction can be generalized to higher dimensions, for example in the 4-dimensional Minkowski space-time we construct a *rough surface* in the following way. We consider a triangle and the first step is to construct a pyramid with the triangle as a base and all angles between the base and the sides of the pyramid equal to the same angle  $\alpha_1$ ; we then iterate the construction decreasing at each step the angle  $\alpha$  between the base

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<sup>1</sup>This notion is unambiguous, as the slope of the rough curve at an endpoint is given by a well-defined limit.

and the sides of the pyramid by a factor  $q < 1/2$  as in the 3-dimensional case. As a result we obtain a nowhere differentiable surface and we define a 3-dimensional fan as the region of  $\Sigma$  bounded by the rough surface and planes perpendicular to the rough surface passing through the sides of the initial triangle.

**Theorem 2** *Let  $b$  be a rough curve and  $F$  the corresponding fan. Then the set of points of  $F$  that are centers of circles tangent to  $b$  in at least two points of  $b$  is dense in the interior of the fan  $F$ .*

*Proof:*

Each point of  $F$  is the center of a circle tangent to  $b$  at at least one point. If the claim of the theorem were false, then there would exist a disc  $D(x, R)$  with nonempty interior with the property that every point  $a \in \text{int } D$  is the center of a circle tangent to the rough curve at exactly one point.

1. A vertex point cannot be a point of tangency of any circle with center in  $\text{int } F$ .
2. By construction the set of vertices of  $b$  is dense in  $b$ . Thus the complement of the set of vertices in  $b$  is totally disconnected (i.e. only one-element subsets are connected).

Let us consider a map  $P$  from the disc to  $b$  that assigns to every point  $y$  of  $D$  a point on  $b$  that is tangent to the circle centered at  $y$ . By assumption this point is unique and thus the map is well-defined.

Let us show that the map  $P$  is continuous. It is enough to prove that if  $a_n \rightarrow a$  then  $P(a_n) \rightarrow P(a)$ . As  $b$  is compact,  $P(a_n)$  has a subsequence that converges to a point  $c$  on  $b$ . Since the distance  $d(a_n, P(a_n))$  is continuous on  $D$  we have  $d(c, a) = d(a, P(a))$ . Hence  $c$  is a tangency point of a circle centered at  $a$  and consequently  $c = P(a)$ .

By the Darboux theorem the image  $P(D(x, R))$  is connected and by 1. and 2. above, it is a one-point set. It then follows that  $R = 0$  which is a contradiction. **QED**

The above theorem generalizes to the 3-dimensional case. In the case of a 3-dimensional fan  $F$  there exists a dense subset of  $F$  such that every ball with the center in this subset has at least two tangency points to the rough

surface. All steps of the proof of Theorem 2 carry over to this case in the natural way.

Let  $\mathcal{H}$  be the set of Cauchy horizons arising from compact convex sets  $K \subset \Sigma$ . The topology on  $\mathcal{H}$  is induced by the Hausdorff distance on the set of compact and convex regions  $K$ .

**Theorem 3** *Let  $\mathcal{H}$  be the set of future Cauchy horizons  $H^+(K)$  where  $K$  are compact and convex regions of  $\Sigma$ . The subset of densely nondifferentiable horizons is dense in  $\mathcal{H}$ .*

*Proof:*

Any compact and convex region  $K$  can be approximated in the sense of Hausdorff distance by a (sequence of) convex polygons contained in  $K$ . Each of the vertex angles of such a polygon is strictly less than  $\pi$ . Over each side of the polygon we construct a rough curve in such a way that the fans corresponding to the rough curves cover the polygon. This is always possible, since we may choose the starting angle  $\alpha_1$  in the rough curve's construction to obey the condition

$$\phi + \frac{2\alpha_1}{1-q} < \pi, \quad (2)$$

where  $\phi$  is the largest vertex angle of the original polygon. When  $\alpha_1$  decreases to 0 the rough-edged polygon converges to the original polygon in the sense of Hausdorff topology. **QED**

It is clear that the above theorem generalizes to higher dimensions.

## 4 Some examples of densely nondifferentiable horizons

In this Section we show that the construction of the previous Section implies the existence of densely nondifferentiable Cauchy horizons of partial Cauchy surfaces and also the existence of black hole event horizons.

**Definition 4** *A partial Cauchy surface  $S$  is a connected, acausal, edgeless  $n-1$  dimensional submanifold of  $(M, g)$ .*

**Example 1:** *A rough wormhole.*

Let  $R^{3,1}$  be the 4-dimensional Minkowski space-time and let  $K$  be a compact subset of the surface  $\{t = 0\}$  such that its Cauchy horizon is nowhere differentiable in the sense of the construction given in Section 3. We consider a space-time obtained by removing the complement of the interior of the set  $K$  in the surface  $t = 0$  from the Minkowski space-time. Let us consider the partial Cauchy surface  $S = \{t = -1\}$ . The future Cauchy horizon of  $S$  is the future Cauchy horizon of set  $K - \text{edge}(K)$ , since  $\text{edge}(K)$  has been removed from the space-time. Thus the future Cauchy horizon is nowhere differentiable and it is generated by past-endless null geodesics. The interior of the set  $K$  can be thought of as a “wormhole” that separates two “worlds”, one in the past of surface  $\{t = 0\}$  and one in its future.

**Example 2:** *A transient black hole.*

Let  $R^{3,1}$  be the 4-dimensional Minkowski space-time and let  $K$  be a compact subset of the surface  $\{t = 0\}$  such that its *past* Cauchy horizon is nowhere differentiable in the sense of the construction given in Section 3. We consider a space-time obtained by removing from Minkowski space-time the closure of the set  $K$  in the surface  $t = 0$ . Let us consider the event horizon  $E := J^-(\mathcal{J}^+)$ . The event horizon  $E$  coincides with  $H^-(K) - \text{edge}(K)$  and thus it is not empty and nowhere differentiable. The event horizon disappears in the future of surface  $\{t = 0\}$  and thus we can think of the black hole (i.e. the set  $B := R^{3,1} - J^-(\mathcal{J}^+)$ ) in the space-time as “transient”.

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## References

- [1] Beem, J. K.: Causality and Cauchy Horizons. *Gen. Rel. and Grav.* **27**, 93-108 (1995)

- [2] Beem, J. K., Ehrlich, P. E., Easley, K. L.: *Global Lorentzian Geometry, Second Edition*, Pure and Appl. Math. vol. 202, Marcel Dekker, New York, 1996
- [3] Beem , J. K., Królak, A.: Censorship and Cauchy Horizons. In: Brown, J. D., Chu, M. T., Ellison, D. C, Plemons, R. J. (eds.) Cornelius Lanczos International Centenary Conference. Proceedings, Raleigh, 1993, pp. 515-517. SIAM, Philadelphia, 1994
- [4] Chruściel, P. T., Galloway, G.: Horizons Non-Differentiable on a Dense Set, Commun. Math. Phys. **193**, 449-470 (1998).
- [5] Beem , J. K., Królak, A.: Cauchy Horizon Endpoints and Differentiability, J. Math. Phys. **39**, 6001-6010 (1998).
- [6] Chruściel, P. T., Isenberg, J.: On the stability of differentiability of Cauchy horizons, Commun. Anal. Geom. **5**, 249–277 (1997)
- [7] Hawking, S. W.: Black holes in general relativity. Commun. Math. Phys. **25**, 152-166 (1972)
- [8] Hawking, S. W.: Chronology Protection Conjecture. Phys. Rev. **D46**, 603-611 (1992)
- [9] Hawking, S. W., Ellis, G. F. R.: *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge 1973
- [10] Królak, A.: Black holes and weak cosmic censorship. Gen. Rel. and Grav. **16**, 365-373 (1984)
- [11] Królak, A.: Towards the proof of cosmic censorship hypothesis. Class. Quantum Grav. **3**, 267 (1986).
- [12] Misner, C. W., Thorne, K. S., Wheeler, J. A.: *Gravitation*, W. H. Freeman, San Francisco, 1973
- [13] O'Neill, B. A.: *Semi-Riemannian geometry*, Pure and Appl. Math. vol. 103, Academic Press, New York, 1983